

# Gauge principle and variational formulation for ideal fluids with reference to translation symmetry

Tsutomu Kambe

*IDS, Higashi-yama 2-11-3, Meguro-ku, Tokyo 153-0043, Japan*

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## Abstract

Following the gauge principle in the field theory of physics, a new variational formulation is presented for flows of an ideal fluid. In the present gauge-theoretical analysis, it is assumed that the field of fluid flow is characterized by a translation symmetry (group) and in addition that the fluid itself is a material in motion characterized thermodynamically by mass density and entropy (per unit mass). Local gauge transformation in the present case is *local Galilean transformation* (without rotation) which is a subgroup of a generalized local Galilean transformation group between non-inertial frames. In complying with the requirement of local gauge invariance of Lagrangians, a gauge-covariant derivative with respect to time is defined by introducing a gauge term. Galilean invariance requires that the covariant derivative should be the *convective* derivative, i.e. the so-called Lagrange derivative. Using this gauge-covariant operator, a free-field Lagrangian and Lagrangians associated with gauge fields are defined under the gauge symmetry. Euler's equation of motion is derived from the action principle. Simultaneously, the equation of continuity and equation of entropy conservation are derived from the variational principle. It is found that general solution thus obtained is equivalent to the classical Clebsch solution. If entropy of the fluid is non-uniform, the flow will be rotational. However, if the entropy is uniform throughout the space (i.e. homentropic), then the flow field reduces to that of a potential flow. Discussions are given on the issue. From the gauge invariance with respect to translational transformations, a differential conservation law of momentum is deduced as Noether's theorem.

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*E-mail address:* [kambe@ruby.dti.ne.jp](mailto:kambe@ruby.dti.ne.jp).

## 1. Introduction

*Fluid mechanics* is a field theory in Newtonian mechanics, i.e. the *field theory of mass flows* subject to Galilean transformation. In the theory of *gauge field*, a guiding principle is that laws of physics should be expressed in a form that is independent of any particular coordinate system. In the gauge theory of particle physics (Weinberg, 1995/1996; Frankel, 1997; Aitchison and Hey, 1982), a free-particle Lagrangian is first defined for a charged particle in such a way as having an invariance under the Lorentz transformation. Next, a gauge principle is applied to the Lagrangian, requiring it to have a *symmetry*, i.e. the gauge invariance. In particular, requirement of *local* gauge invariance implies existence of a new gauge field which is associated with the electromagnetic field.

There are obvious differences between the fluid-flow field and the quantum field. Firstly, the field of fluid flow is non-quantum, which however causes no problem since the gauge principle is independent of the quantization principle. In addition, the fluid flow is subject to the Galilean transformation instead of the Lorentz transformation. This is not an obstacle because the former is a limiting transformation of the latter as the relative ratio of flow velocity to the light speed tends to an infinitesimal quantity. Thirdly, relevant gauge groups should be different. Certainly, we have to find appropriate gauge groups for fluid flows. A translation group and a rotation group would be such groups relevant to fluid flows.

Here, we seek a formulation of fluid flows which has a formal equivalence with the gauge theory in the electromagnetism or quantum field theory. Following the scenario of the gauge principle, we define at the outset a Galilei-invariant Lagrangian for a system of *point masses* which is known to have *global* gauge invariance (in *Mechanics* by Landau and Lifshitz, 1976). We try to extend it to fluid flows. For a continuous field such as a fluid, in addition to the global symmetry, *local* gauge invariance of Lagrangian is required. This is satisfied by introducing a new gauge term into time derivative term. Precise expressions of the global and local invariance will be presented below, and explicit forms of the Lagrangian will be given at each step of derivation.

In the present paper, we try to apply the above concept to the formulation of flows of an ideal fluid. This approach results in a unified description of flow fields and a reformulation on the basis of the gauge principle, which discloses some new aspects. Usually, the convective derivative of the velocity (i.e. the Lagrange derivative) is written down intuitively for the acceleration of a material particle and taken as an identity without relying on any physical or mathematical principle. In the present formulation, the same convective derivative is derived as the *covariant derivative* in the framework of the gauge theory, which is an essential building block of the theory. Previous papers (Kambe, 2003a,b) tried to apply the concept of the gauge symmetry of rotational transformations to fluid flows, and found that the vorticity is the gauge field associated with the rotational symmetry of fluid flows.

According to the traditional variational formulation referred to as Eulerian description, if the fluid is *homentropic* (i.e. the fluid entropy is uniform throughout space), the action principle of an ideal fluid results in potential flows. It is generally understood that, even in such a homentropic fluid, it should be possible to have rotational flows. In fact, this is a long-standing problem (Serrin, 1959; Lin, 1963; Seliger and Whitham, 1968; Bretherton, 1970; Salmon, 1988). Lin (1963) tried to resolve this difficulty by introducing the *Lin's constraint* as a side condition, imposing invariance of Lagrangian particle coordinates along particle trajectories. The constraints for the variation are formulated by using Lagrange multipliers (functions of positions), which are called as potentials. In addition, the continuity equation and isentropic condition are also taken into account by using Lagrange multipliers, where the *isentropy* means that each fluid particle keeps its entropy value along its trajectory but that the fluid is not necessarily homentropic.

However, physical significance of those potentials introduced as the Lagrange multipliers is not clear. Mysteriously, the Lagrange multiplier for the continuity equation becomes the velocity potential for flows of a homentropic fluid (without the Lin's constraint).

The present gauge theory for fluid flows provides us a crucial key to resolve the above issues. This is the main theme of the present paper. Among the two symmetries of flows mentioned above, the present paper concentrates on the translation symmetry, in order to focus on the theoretical framework of the gauge theory applied to fluid flows and show its powerfulness. It is found that general solution in this formulation is equivalent to the classical Clebsch solution. The rotational symmetry will be considered elsewhere in future, but its preliminary approach is already given in Kambe (2003a,b). Some consideration is given to scaling symmetry of the present system in Appendix C.<sup>1</sup>

Similar field-theoretic approach is taken in Jackiw (2002) by applying the ideas of particle physics to fluid mechanics in terms of Hamiltonians of canonical variables and Poisson brackets, both relativistically and nonrelativistically, and extension to supersymmetry is also considered. In this monograph, the nonrelativistic part follows the traditional approach and gauge-theoretic consideration is not given to fluid mechanics. In a Galilean-invariant nonrelativistic case, symmetries of a specific model of the Chaplygin gas with a particular equation of state are studied. In addition to the symmetries with respect to space–time translations and rotations of Galilean group, this model is shown to have a time rescaling symmetry and a space–time mixing symmetry. In addition, the Clebsch solution is featured to represent a vorticity field  $\omega = \nabla \times \mathbf{v}$  in terms of three scalar functions and investigate the helicity  $H$  (Chern–Simon term) for the velocity  $\mathbf{v}$ , where  $H$  is defined by a 3D space integral of  $\omega \cdot \mathbf{v}$  (see Appendix B for its definition).

Some backgrounds of the present theory are reviewed in the followings.

### 1.1. Lagrangian and symmetries

Lagrange already knew the conservation of momentum as a result of translation symmetry, which is now understood as a global symmetry of Lagrangian. This property is generalized as the Noether theorem (Noether, 1918; Soper, 1976), and it is commonly known that all conservation laws are derived from invariances of the Lagrangian under transformations, i.e. its symmetries. The symmetries considered in mechanics literatures are mostly global, i.e. independent of points in the space. The local symmetry which we are going to investigate here from the viewpoint of the gauge theory is not considered so far, at least explicitly in mechanics.

Schutz and Sorkin (1977) verified that any variational principle for an ideal fluid that leads to the Euler equation of motion must be constrained. This is related to the property of fluid flows that the energy (including the mass energy) of a fluid *at rest* can be changed by adding entropy or adding particles (i.e. changing density) without violating the framework of the variational principle. In addition, adding a uniform velocity to a uniform-flow state is again another state of uniform flow.

With regard to the gauge field, we must find an appropriate Lagrangian. In the case of the system of charged particles and electromagnetic field, the total Lagrangian density consists of three parts:  $L_p + L_{pf} + L_f$ . The part of Lagrangian associated with charged particles is  $L_p$ , while the  $L_f$  represents the Lagrangian that depends on the electromagnetic fields only, i.e. the Lagrangian in the absence of charged particles, where  $L_{pf}$  is the part of interaction between the particle and fields. In order to obtain the equations of motion of particles by the variational principle (e.g. *The Classical Theory of Fields* by Landau and

<sup>1</sup> The Appendix C is added in the revised version as a response of the author to a comment of a referee.

Lifshitz, 1975), we must assume the field to be given (so that it is kept fixed) and vary only the trajectory of the particles, while in order to find the field equations, we must assume the motion of the charges to be given and vary only the field potentials.

Suppose that we have a gauge invariance, i.e. we have a symmetry group of transformations of its Lagrangian. Corresponding to the gauge invariance, the Noether theorem leads to a conservation law (Soper, 1976; Schutz and Sorkin, 1977). In fact, the gauge symmetry with respect to the translation group results in the differential conservation law of momentum, while the symmetry with respect to the rotation group results in the conservation of angular momentum (Kambe, 2003b; 2004). The same is true in the theory of elasticity (Maugin, 1993; Marsden and Hughes, 1993). In addition, the Lagrangian has an internal symmetry with respect to particle coordinates. In the study of elasticity of inhomogeneous materials (Maugin, 1993), the variational formulation with respect to the particle coordinates is termed as the inverse-motion description, and the invariance property of the Lagrangian under infinitesimal rotation of the material frame is investigated.

### 1.2. Gauge invariance and symmetry of fluid flows

Gauge theory of an electromagnetic system is closely associated with the invariance of the field under gauge transformation of electromagnetic potentials. In fact, the electric and magnetic field vectors are represented as  $\mathbf{E} = -\partial_t \mathbf{A} - \nabla \phi$  and  $\mathbf{M} = \nabla \times \mathbf{A}$ , respectively, where  $\phi$  and  $\mathbf{A}$  are scalar and vector potentials respectively, and  $\partial_t = \partial/\partial t$ . The fields  $\mathbf{E}$  and  $\mathbf{M}$  are unchanged by the transformations:  $\phi \rightarrow \phi - \partial_t f$  and  $\mathbf{A} \rightarrow \mathbf{A} + \nabla f$  for an arbitrary differentiable scalar function  $f(\mathbf{x}, t)$  of position vector  $\mathbf{x}$  and time  $t$ .

In regard to potential flows of an ideal fluid, it is interesting and important to recall that there is a similar invariance under a (gauge) transformation of velocity potential  $\varphi$  in fluid mechanics where the velocity field is represented as  $\mathbf{v} = \nabla \varphi$ , although this is not referred to as a gauge invariance in conventional fluid mechanics. In fact, a potential flow of a homentropic fluid has an integral of motion expressed by  $\frac{1}{2}v^2 + h + \partial_t \varphi = F(t)$ , where  $h$  is the enthalpy of the fluid and  $F(t)$  an arbitrary differentiable scalar function of time  $t$ . It is evident that the velocity  $\mathbf{v}$  and the integral are unchanged by the transformations:  $\varphi \rightarrow \varphi + f(t)$  and  $F \rightarrow F + \partial_t f$  for another arbitrary scalar function  $f(t)$ .

The symmetry group of flows we are going to consider is the *translation group*. It is assumed that the Lagrangian is gauge-invariant with respect to transformations of parallel (i.e. non-rotational) translation, both global and local. Then, from the variational principle applied to the Lagrangian, we will obtain Euler's equation of motion of an ideal fluid. The flow field is rotational if the entropy is non-uniform in space. According to the non-dissipative nature of ideal fluids by definition, the motion is *isentropic* in an ideal fluid. However, if the fluid is homentropic, the flow is found to be irrotational. Namely, as far as the translational (non-rotational) symmetry is concerned, the equation of motion obtained from the action principle will be that of potential flows for a homentropic fluid. It is well-known that flows of a superfluid in the degenerate ground state are irrotational (e.g. *Fluid Mechanics* by Landau and Lifshitz, 1987; Pethick and Smith, 2002).

A successful formulation of flows of an incompressible ideal fluid is the geometrical theory based on the Riemannian geometry and Lie group theory (Arnold, 1966, 1978; Kambe, 2004, Chapter 8, for its review). Euler's equation of motion is derived as a geodesic equation over the manifold of a group of volume-preserving diffeomorphisms with the Riemannian metric defined by the kinetic energy, and the behaviors of the geodesics are controlled by Riemannian curvature tensors. The gauge group is the group of volume-preserving diffeomorphisms. However, this approach is more mathematical in the sense that

local translation and local rotation are not separated. Here we try to separate the two in order to get insight into physics of flows.

### 1.3. System of point masses

Suppose that we are given a system of  $n$  point masses  $m_k$  ( $k = 1, \dots, n$ ) whose positions are denoted by  $\mathbf{x}_1 = (q^1, q^2, q^3), \dots, \mathbf{x}_n = (q^{3n-2}, q^{3n-1}, q^{3n})$ . Their velocities are written by  $\mathbf{v}_j = (v_j^1, v_j^2, v_j^3) = (q_t^{3j-2}, q_t^{3j-1}, q_t^{3j})$  for  $j = 1, \dots, n$ . We consider a Lagrangian  $L$  of the form

$$L = L[q, q_t], \quad (1)$$

which depends on the coordinates  $q = q(t) = (q^i)$  and the velocities  $q_t = \partial_t q = (q_t^i)$  for  $i = 1, 2, \dots, 3n$ . The Lagrangian  $L$  describes a dynamical system of  $3n$  degrees of freedom. The action  $I$  is defined by  $I = \int_{t_0}^{t_1} L[q, q_t] dt$ . The principle of least action, i.e. the Hamilton principle, is given by

$$\delta I = \int_{t_0}^{t_1} \delta L[q, q_t] dt = 0 \quad (2)$$

together with fixed values of  $q$  and  $q_t$  at both ends  $t_0$  and  $t_1$  of time  $t$ . This results in the Euler–Lagrange equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial q_t^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad (3)$$

where the variation to a reference trajectory  $q(t)$  is written as (say)  $q'(t, \varepsilon) = q(t) + \varepsilon \xi(t)$  and  $q_t' = \dot{q}(t) + \varepsilon \dot{\xi}(t)$  with a virtual displacement  $\xi(t)$  vanishing at  $t_0$  and  $t_1$ .

If the Lagrangian is given by the following form for  $n$  masses  $m_j$  ( $j = 1, \dots, n$ ),

$$L_f = \frac{1}{2} \sum_{j=1}^n m_j \langle \mathbf{v}_j, \mathbf{v}_j \rangle, \quad (4)$$

$L_f$  is called a free-particle Lagrangian, where  $\langle \mathbf{v}_j, \mathbf{v}_j \rangle = \sum_{k=1}^3 v_j^k v_j^k$  is the inner product, because the above equation (3) results in the equation of *free motion*:

$$\partial_t \left( \frac{\partial L_f}{\partial v_j^k} \right) = \partial_t p_j^k = 0, \quad \frac{\partial L_f}{\partial v_j^k} \equiv p_j^k = m_j v_j^k,$$

i.e. the momentum vector  $\mathbf{p}_j = m_j \mathbf{v}_j$  is constant.

### 1.4. Global invariance and conservation law

Let us consider a translational transformation of parallel displacement in which every particle in the system is moved by the same amount  $\xi$ , i.e. the position vector  $\mathbf{x}_j$  is replaced by  $\mathbf{x}_j + \xi$ , where  $\xi = (\xi^1, \xi^2, \xi^3)$

is an arbitrary constant infinitesimal vector.<sup>2</sup> Resulting variation of the Lagrangian  $L$  is denoted by  $\delta L$  (assuming  $\xi$  a function of  $t$ ). Then, we have

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial q_t} \delta q_t = \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial q_t^i} \partial_t (\delta q^i) \\ &= \left( \frac{\partial L}{\partial q^i} - \partial_t \left( \frac{\partial L}{\partial q_t^i} \right) \right) \delta q^i + \partial_t \left( \frac{\partial L}{\partial q_t^i} \delta q^i \right),\end{aligned}\quad (5)$$

where  $\delta q^{3j-3+k} = \delta x_j^k = \xi^k$  for  $j = 1, \dots, n$  and  $k = 1, 2, 3$ . (In this transformation, the velocities of the particles remain fixed since  $\delta q_t = \partial_t (\delta q) = 0$ .) The first term of (5) vanishes owing to the Euler–Lagrange equation (3). Thus,

$$\delta L = \partial_t \left( \frac{\partial L}{\partial q_t^i} \delta q^i \right).\quad (6)$$

When the displacement  $\xi$  is a constant vector for all  $x_j$  ( $j = 1, \dots, n$ ) like in the present case, the transformation is called *global*. Requiring that the Lagrangian is invariant under this transformation, i.e.  $\delta L = 0$ , we have

$$\partial_t \left( \frac{\partial L}{\partial q_t^i} \delta q^i \right) = \xi^k \sum_{k=1}^3 \partial_t \left( \sum_{j=1}^n \frac{\partial L}{\partial v_j^k} \right) = 0.$$

Since  $\xi^k$  ( $k = 1, 2, 3$ ) are arbitrary, we obtain

$$\sum_{j=1}^n \frac{\partial L}{\partial v_j^1} = \text{const}, \quad \sum_{j=1}^n \frac{\partial L}{\partial v_j^2} = \text{const}, \quad \sum_{j=1}^n \frac{\partial L}{\partial v_j^3} = \text{const}.$$

Thus the three components of the total momentum are conserved. This is the *Noether theorem* for the global invariance. It is well-known that Newton's equation of motion is invariant with respect to Galilean transformation, i.e. a transformation between two inertial frames of reference in which one frame is moving with a constant velocity  $U$  relative to the other. The Galilean transformation is a sequence of *global* translational gauge transformations with respect to the time parameter  $t$ .

The global invariance with respect to translational transformations is associated with the homogeneity of space, while global invariance with respect to rotational transformations is associated with the isotropy (Landau and Lifshitz, 1976, Sections 7 and 9).

## 2. Gauge transformations

We investigate how the Lagrangian of the form (1), or (4), of discrete systems must be modified for a system of fluid flows characterized by a continuous distribution of mass. According to the principle of gauge invariance, we consider gauge transformations in general, which are both global and local (Weinberg, 1995; Frankel, 1997; Utiyama, 1978). In later sections we will concentrate on a particular gauge transformation of flow fields.

<sup>2</sup> This is a global gauge transformation different from the variational principle considered in the previous Section (1.3).

Concept of local transformation is a generalization of the global transformation. When we consider local gauge transformation, the physical system under consideration must be modified so as to allow us to consider a continuous field by extending the original discrete system. We replace the discrete variables  $q^i$  by continuous parameters  $\mathbf{a} = (a^1, a^2, a^3)$  to represent continuous distribution of particles in a sub-space  $M$  of three-dimensional Euclidean space  $E^3$ . Spatial position  $\mathbf{x} = (x^1, x^2, x^3)$  of each massive particle of the name tag  $\mathbf{a}$  (Lagrange parameter) is denoted by  $\mathbf{x} = \mathbf{x}_a(t) \equiv \mathbf{X}(\mathbf{a}, t) = (X^k(\mathbf{a}, t))$ , a function of  $\mathbf{a}$  as well as time  $t$ . Conversely, the particle occupying the point  $\mathbf{x}$  at a time  $t$  is denoted by  $\mathbf{a}(\mathbf{x}, t)$ .

### 2.1. Continuous field and global invariance

Now, we consider a continuous distribution of mass (i.e. *fluid*) and its motion. The Lagrangian (1) or (4) must be modified to the following integral form:

$$L = \int \mathcal{L}(q, q_t) d^3\mathbf{x}, \quad (7)$$

where  $\mathcal{L}$  is a Lagrangian density. Suppose that an infinitesimal transformation is expressed by

$$\left. \begin{aligned} q = \mathbf{x} &\rightarrow q' = \mathbf{x} + \delta\mathbf{x}, & \delta\mathbf{x} &= \boldsymbol{\xi}(\mathbf{x}, t), \\ q_t = \mathbf{v} &\rightarrow q'_t = \mathbf{v} + \delta\mathbf{v}, & \delta\mathbf{v} &= \partial_t(\delta\mathbf{x}), \end{aligned} \right\} \quad (8)$$

where  $\mathbf{v} = \partial_t \mathbf{X}(\mathbf{a}, t)$ , and the vector function  $\boldsymbol{\xi}(\mathbf{x}, t)$  is an arbitrary differentiable variation field. Resulting variation of the Lagrangian density  $\mathcal{L}(\mathbf{x}, \mathbf{v})$  is

$$\delta\mathcal{L} = \left( \frac{\partial\mathcal{L}}{\partial\mathbf{x}} - \partial_t \left( \frac{\partial\mathcal{L}}{\partial\mathbf{v}} \right) \right) \cdot \delta\mathbf{x} + \partial_t \left( \frac{\partial\mathcal{L}}{\partial\mathbf{v}} \cdot \delta\mathbf{x} \right). \quad (9)$$

This does not vanish in general owing to the arbitrary function  $\delta\mathbf{x} = \boldsymbol{\xi}(\mathbf{x}, t)$  depending on time  $t$ . In fact, assuming the Euler–Lagrange equation  $\partial\mathcal{L}/\partial\mathbf{x} - \partial_t(\partial\mathcal{L}/\partial\mathbf{v}) = 0$  (an extended form of (3)), we obtain

$$\delta\mathcal{L} = \partial_t \left( \frac{\partial\mathcal{L}}{\partial\mathbf{v}} \cdot \delta\mathbf{x} \right) = \partial_t \left( \frac{\partial\mathcal{L}}{\partial\mathbf{v}} \right) \cdot \boldsymbol{\xi} + \frac{\partial\mathcal{L}}{\partial\mathbf{v}} \cdot \partial_t \boldsymbol{\xi}. \quad (10)$$

In the global transformation of  $\boldsymbol{\xi} = \text{const}$ , we have  $\partial_t \boldsymbol{\xi} = 0$ . Then, global invariance of the Lagrangian ( $\delta L = 0$ ) for arbitrary constant  $\boldsymbol{\xi}$  requires

$$\partial_t \int \frac{\partial\mathcal{L}}{\partial\mathbf{v}} d^3\mathbf{x} = 0.$$

This states the conservation of total momentum defined by  $\int (\partial\mathcal{L}/\partial\mathbf{v}) d^3\mathbf{x}$ . The same result for the global transformation ( $\boldsymbol{\xi} = \text{const}$  and  $\delta\mathbf{v} = \partial_t \boldsymbol{\xi} = 0$ ) can be obtained directly from (7) since

$$\delta L = \int \boldsymbol{\xi} \cdot \frac{\partial\mathcal{L}}{\partial\mathbf{x}} d^3\mathbf{x} = \boldsymbol{\xi} \cdot \int \partial_t \left( \frac{\partial\mathcal{L}}{\partial\mathbf{v}} \right) d^3\mathbf{x} = 0, \quad (11)$$

by using the above Euler–Lagrange equation. In the local transformation, however, the variation field  $\boldsymbol{\xi}$  depends on time  $t$  and space point  $\mathbf{x}$ , and the variation  $\delta L = \int \delta\mathcal{L} d^3\mathbf{x}$  does not vanish in general.



### 2.2. Covariant derivative

According to the gauge principle (e.g. Weinberg, 1995/1996; Aitchison and Hey, 1982), non-vanishing of  $\delta L$  is understood as meaning that a *new field*  $G$  must be taken into account in order to achieve *local gauge invariance* of the Lagrangian  $L$ . To that end, we try to replace the partial time derivative  $\partial_t$  in (9) by a *covariant derivative*  $D_t$ , where the derivative  $D_t$  is defined by

$$D_t = \partial_t + G, \tag{12}$$

with  $G$  being a gauge field (an operator). The time derivatives  $\partial_t \zeta$  and  $\partial_t q$  are replaced by

$$D_t q = \partial_t q + G q, \quad D_t \zeta = \partial_t \zeta + G \zeta. \tag{13}$$

Correspondingly, we assume that the Lagrangian  $L_f$  of (4) is replaced by

$$L_f = \int \mathcal{L}_f(q, q_t, G) d^3x \equiv \frac{1}{2} \int \langle D_t \mathbf{x}_a, D_t \mathbf{x}_a \rangle d^3a, \tag{14}$$

where  $d^3a = \rho d^3x$  denotes the mass (in place of  $m_k$ ) in a volume element  $d^3x$  of the  $x$ -space with  $\rho$  the mass-density.<sup>3</sup> In dynamical systems like the present case, the *time derivative* is the primary object to be considered in the analysis of local gauge transformation. This is consistent with the invariance noted in the item (b) of the introduction. The action is defined by

$$I = \int_{t_0}^{t_1} L_f[q, q_t] dt = \int_{t_0}^{t_1} dt \int_M d^3x \mathcal{L}_f(q, q_t, G), \tag{15}$$

where  $M$  is a bounded space of  $E^3$ , and

$$\mathcal{L}_f = \frac{1}{2} \rho \langle D_t q, D_t q \rangle = \frac{1}{2} \rho \langle D_t \mathbf{x}_a, D_t \mathbf{x}_a \rangle. \tag{16}$$

We will consider below how the Lagrangian  $L_f$  is invariant under local infinitesimal transformations.

### 2.3. Gauge group of flow fields: translational transformations

It was seen that the Lagrangian (4) has a global symmetry with respect to the translational transformations (and possibly with respect to rotational transformations). A family of translational transformations is a *group* of transformations,<sup>4</sup> i.e. a translation group. Lagrangian defined by (14) for a continuous field has the same properties globally, inheriting from the discrete system of point masses. It is a primary concern of the present analysis to investigate whether the system of fluid flows satisfies local invariance. We consider parallel translations (without local rotation), where the coordinate  $q^i$  is regarded as the Cartesian space coordinate  $x^k$  ( $k$ th component), and  $q_t^i$  is taken as a velocity component  $v^k = \partial_t X^k(\mathbf{a}, t)$ .

Suppose that we have a differentiable function  $f(x)$ . Its variation by an infinitesimal translation  $x \rightarrow x + \zeta$  is given by  $\delta f = \zeta \partial_x f$  where  $\partial_x \equiv \partial/\partial x$  is regarded as a translation operator with a parameter  $\zeta$ .

<sup>3</sup> Here the Lagrangian coordinates  $\mathbf{a} = (a, b, c)$  are defined so as to represent the *mass coordinate*. Using the Jacobian of the map  $x \mapsto \mathbf{a}$  defined by  $J = \partial(\mathbf{a})/\partial(x)$ , we have  $d^3a = J d^3x$ , where  $J$  is  $\rho$ .

<sup>4</sup> A family of translational transformations is a *group* characterized with a product law of two elements of the group together with existence of an identity element and a unique inverse operation. If the product law is commutative, the group is called a commutative group, or an *Abelian* group.



The operator of the parallel translation is denoted by  $T_k = \partial/\partial x^k = \partial_k$ , ( $k = 1, 2, 3$ ). An arbitrary translation is represented by  $\xi^k T_k$  ( $\equiv \sum_{k=1}^3 \xi^k T_k$ ) with  $\xi^k$  an infinitesimal parameter. For example, a variation of  $x^l$  is given by

$$\delta x^l = (\xi^k T_k) x^l = \xi^k \delta_k^l = \xi^l.$$

The generators  $T_k$  are commutative, i.e. the commutator is given by  $[T_k, T_l] = \partial_k \partial_l - \partial_l \partial_k = 0$ , i.e. Abelian. Hence the structure constants are all zero in the translational transformation.

### 3. Translational transformation and gauge field

We study invariance properties under local Galilean (gauge) transformations. From it we deduce the form of the covariant derivative  $D_t = \partial_t + G$  and the gauge operator  $G$  which satisfy local invariance. Next, we propose Lagrangians invariant under local gauge transformations and a possible Lagrangian ruling the gauge field.

#### 3.1. Local Galilean invariance

Suppose that we have a velocity field  $v(x, t)$  of an ideal fluid in a flat space  $E^3$ . We consider the following infinitesimal transformation:

$$\mathbf{x}'(x, t) = \mathbf{x} + \boldsymbol{\xi}(x, t), \quad (17)$$

(local gauge transformation),<sup>5</sup> without influencing the velocity field  $v(x, t)$  in the (assumed) inertial frame  $F_M$  where the point  $\mathbf{x} \in M \subset E^3$  is expressed in the reference frame  $F_M$ . This is regarded as a subgroup of generalized local Galilean transformation group between non-inertial frames. In fact, the transformations (17) is understood to mean that the coordinate  $\mathbf{x}$  of a fluid particle at  $\mathbf{x} = \mathbf{x}_a(t)$  is transformed to the new coordinate  $\mathbf{x}'$  of  $F'_M$ , which is given by  $\mathbf{x}' = \mathbf{x}'_a(\mathbf{x}_a, t) = \mathbf{x}_a(t) + \boldsymbol{\xi}(\mathbf{x}_a, t)$ . Therefore, its velocity  $\mathbf{v} = (d/dt)\mathbf{x}_a(t)$  is transformed to the following representation:

$$\begin{aligned} \mathbf{v}'(\mathbf{x}', t) &\equiv \frac{d}{dt}\mathbf{x}'_a = \frac{d}{dt}(\mathbf{x}_a(t) + \boldsymbol{\xi}(\mathbf{x}_a, t)) = \mathbf{v}(\mathbf{x}_a, t) + \frac{d}{dt}\boldsymbol{\xi}(\mathbf{x}_a(t), t), \\ \frac{d}{dt}\boldsymbol{\xi}(\mathbf{x}_a(t), t) &= \partial_t \boldsymbol{\xi} + \mathbf{v} \cdot \nabla \boldsymbol{\xi} \quad (\text{at } \mathbf{x} = \mathbf{x}_a(t)). \end{aligned} \quad (18)$$

This is interpreted as follows: the local coordinate origin is displaced by  $-\boldsymbol{\xi}$  with the axes of the local frame moving (without rotation) with the velocity  $-(d/dt)\boldsymbol{\xi}$  in accelerating motion (non-inertial frame).<sup>6</sup> This implies that the velocity  $\mathbf{v}(\mathbf{x})$  is transformed locally at  $\mathbf{x}$  as  $\mathbf{v}'(\mathbf{x}') = \mathbf{v}(\mathbf{x}) + (d/dt)\boldsymbol{\xi}$ , since the local frame is moving with the velocity  $-(d/dt)\boldsymbol{\xi}$ . Note that the points  $\mathbf{x}$  and  $\mathbf{x}'$  are the same points with respect to the space  $F_M$ .

<sup>5</sup> Bold letters denote vectors of three components, e.g.  $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)$ .

<sup>6</sup> Usual Galilean transformation defined by  $\boldsymbol{\xi} = -U t$  is global, where the relative velocity is a constant vector  $U$ .

In view of the transformation defined by  $t' = t$  and  $\mathbf{x}' = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$ , the time derivative and spatial derivatives are transformed as

$$\partial_t = \partial_{t'} + (\partial_t \boldsymbol{\xi}) \cdot \nabla', \quad \nabla' = (\partial'_k), \quad \partial'_k = \partial / \partial' x^k, \quad (19)$$

$$\partial_k = \partial'_k + \partial_k \xi^l \partial'_l, \quad \nabla = J \nabla', \quad J_k^l = \delta_k^l + \partial_k \xi^l, \quad (20)$$

where  $J = (J_k^l)$ , and the inverse  $J^{-1}$  is assumed to exist with  $\det J > 0$ .

We require that the derivative  $D_t = \partial_t + G(\mathbf{x})$  is *covariant* in the sense that  $D_t = D_{t'}$ , where  $D_{t'} \equiv \partial_{t'} + G'(\mathbf{x}')$  and

$$D_t = \partial_t + G(\mathbf{x}) = \partial_{t'} + (\partial_t \boldsymbol{\xi}) \cdot \nabla' + G(\mathbf{x}), \quad (21)$$

by (19). Eliminating  $\partial_t \boldsymbol{\xi}$  by using (18) in the equality  $D_t = D_{t'}$ , where  $\partial_t \boldsymbol{\xi} = (d/dt)\boldsymbol{\xi} - \mathbf{v} \cdot \nabla \boldsymbol{\xi}$ , we have

$$G'(\mathbf{x}') - G(\mathbf{x}) = (\partial_t \boldsymbol{\xi}) \cdot \nabla' = (\mathbf{v}'(\mathbf{x}') - \mathbf{v}(\mathbf{x})) \cdot \nabla' - (\mathbf{v} \cdot \nabla) \boldsymbol{\xi} \cdot \nabla',$$

since  $(d/dt)\boldsymbol{\xi} = \mathbf{v}'(\mathbf{x}') - \mathbf{v}(\mathbf{x})$  from (18). This is rewritten as

$$G'(\mathbf{x}') - \mathbf{v}'(\mathbf{x}') \cdot \nabla' = G(\mathbf{x}) - \mathbf{v}(\mathbf{x}) \cdot \nabla' - \mathbf{v} \cdot (\nabla \boldsymbol{\xi} \cdot \nabla'),$$

which reduces finally to (by using (20) where  $\partial_k \xi^l \partial'_l = \nabla \boldsymbol{\xi} \cdot \nabla'$ )

$$G' - \mathbf{v}' \cdot \nabla'|_{x'} = G - \mathbf{v} \cdot \nabla|_x. \quad (22)$$

This implies that  $G = \mathbf{v} \cdot \nabla$  and  $G' = \mathbf{v}' \cdot \nabla'|$ .<sup>7</sup> Thus, the following covariance is obtained:

$$D_t|_x = D_{t'}|_{x'} \quad \text{where} \quad D_t = \partial_t + \mathbf{v} \cdot \nabla, \quad D_{t'} = \partial_{t'} + \mathbf{v}' \cdot \nabla', \quad (23)$$

under the transformations (17)–(20).

In terms of the covariant derivative  $D_t$ , we can define the velocity  $\mathbf{v}$  by  $D_t \mathbf{x}$ . In fact, the Lagrange particle coordinate  $\mathbf{a}$  (Section 2) satisfies

$$D_t \mathbf{a} = \partial_t \mathbf{a} + (\mathbf{v} \cdot \nabla) \mathbf{a} = 0, \quad (24)$$

since the particle of the name tag  $\mathbf{a}$  moves with the velocity  $\mathbf{v}$  by definition. Setting as  $\mathbf{x} = \mathbf{X}(\mathbf{a}, t)$  for the particle position, we have

$$\mathbf{v} = D_t \mathbf{X}(\mathbf{a}, t) = \partial_t \mathbf{X}(\mathbf{a}, t) + D_t \mathbf{a} \cdot \nabla_a \mathbf{X} = \partial_t \mathbf{X}(\mathbf{a}, t), \quad (25)$$

by using (24), where  $(\nabla_a \mathbf{X}) = (\partial X^k / \partial a^l)$ . On the other hand, regarding  $\mathbf{x}$  as a field variable, we have  $D_t \mathbf{x} = (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{x} = \mathbf{v}$ , which is consistent with above, defining  $\mathbf{v}$  as the velocity of a fluid particle of name tag  $\mathbf{a}$ . Applying  $D_{t'}$  ( $= D_t$ ) to (17), we have

$$\mathbf{v}' = D_{t'} \mathbf{x}' = D_t (\mathbf{x} + \boldsymbol{\xi}) = \mathbf{v} + D_t \boldsymbol{\xi}.$$

This is consistent with (18). Thus, we have found that

$$G = \mathbf{v} \cdot \nabla, \quad D_t = \partial_t + \mathbf{v} \cdot \nabla. \quad (26)$$

<sup>7</sup> Eq. (22) implies that the right-hand side may be a constant  $c$ . However, non-zero  $c$  leads to non-zero value of  $D_t f = cf$  for a steady uniform field  $f$ . This should be excluded in the present problem. Hence,  $c = 0$ .

Applying the operator  $D_t$  on a scalar function  $f(\mathbf{x}, t)$ , we have

$$D_t f = \partial_t f + \mathbf{v} \cdot \nabla f = \partial_t f + u \partial_x f + v \partial_y f + w \partial_z f, \quad (27)$$

called the *convective* derivative, or *material* derivative. This denotes the rate of change of the value of the function  $f(\mathbf{x}, t)$  when the reference point moves with the velocity  $\mathbf{v}=(u, v, w)$ . It is important to recognize that this implies existence of a background *material* which is moving with the velocity  $\mathbf{v}(x, y, z)$ .

### 3.2. Gauge invariant Lagrangians

By using the explicit expression (26) of  $D_t$ , the Lagrangian  $L_f$  of (14) can be written as<sup>8</sup>

$$L_f = \frac{1}{2} \int_M \langle \mathbf{u}, \mathbf{u} \rangle \rho d^3\mathbf{x} \quad (\mathbf{u} = D_t \mathbf{x}, \quad D_t \equiv \partial_t + \mathbf{u} \cdot \nabla). \quad (28)$$

According to the transformation (17) and (18), this is transformed to

$$L'_f = \frac{1}{2} \int_{M'} \langle \mathbf{u}'(\mathbf{x}'), \mathbf{u}'(\mathbf{x}') \rangle \rho'(\mathbf{x}') d^3\mathbf{x}', \quad (29)$$

where  $M'$  is the transformation of  $M$ .

In the traditional consideration of Galilean invariance, the transformation is defined by  $\xi = Ut$  and  $\mathbf{u}' = \mathbf{u} + \mathbf{U}$ , where  $\mathbf{U}$  is the constant relative velocity between two inertial frames. It appears by naive observation that the  $L_f$  is not invariant by this transformation since  $\langle \mathbf{u}', \mathbf{u}' \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{U} \rangle + \langle \mathbf{U}, \mathbf{U} \rangle$ . This is resolved by returning to the Lorentz invariance from which the Galilean transformation is derived as a limit. This issue is discussed briefly in Appendix A, and its detailed account is given in Kambe (2003b). By the reasoning given in Appendix A, the Lagrangian  $L_f$  must be replaced by  $A_L^{(0)}$ , which is invariant with respect to the above transformation regarded as a limiting form of *relativistic* Lorentz transformation. The above transformation is a sequence of global transformations with the time parameter  $t$ . This is reasonable, since the transformation is regarded as shifting of the coordinate frame without influencing the velocity field  $\mathbf{u}$  and  $\mathbf{x}'$  is equivalent to  $\mathbf{x}$  in the inertial frame  $F_M$ , and the mass  $\rho'(\mathbf{x}') d^3\mathbf{x}'$  should be equivalent to  $\rho(\mathbf{x}) d^3\mathbf{x}$ . Detailed analysis of invariance with respect to local gauge transformations will be carried out in a later section (Section 5.2).

According to the physical derivation (Appendix A and Kambe, 2003b) on the basis of the Lorentz invariance of the Lagrangian, it is found that the Lagrangian  $L_f$  must be supplemented by a Lagrangian  $L_\varepsilon$  defined by

$$L_\varepsilon = - \int_M \varepsilon(\rho, s) \rho d^3\mathbf{x}, \quad (30)$$

where  $\varepsilon(\rho, s)$  is the internal energy,  $\rho$  the fluid density and  $s$  the entropy. It is understood that the background continuous material is characterized by the internal energy  $\varepsilon$  of the fluid, given by a function  $\varepsilon(\rho, s)$  of density  $\rho$  and entropy  $s$  where  $\varepsilon$  and  $s$  are defined per unit mass.<sup>9</sup> For the fields of density  $\rho(\mathbf{x})$  and entropy  $s(\mathbf{x})$ , the Lagrangian  $L_\varepsilon$  is invariant with respect to the gauge transformation (17), since the transformation

<sup>8</sup> Henceforth, we use  $\mathbf{u}$  to denote the velocity since we are considering translational symmetry only.

<sup>9</sup> In thermodynamics, a physical material of a single phase is characterized by two thermodynamic variables such as  $\rho, s$ , etc.

is a matter of the coordinate origin under the invariance of mass:  $\rho(\mathbf{x}) d^3\mathbf{x} = \rho'(\mathbf{x}') d^3\mathbf{x}'$  and the coordinate  $\mathbf{x}$  does not appear explicitly. Detailed analysis of invariance will be given in a later section.

According to the scenario of the gauge principle, an additional Lagrangian (called a *kinetic* term, Frankel, 1997) is to be defined in connection with the background field (the material field *in motion* in the present context), in order to get non-trivial field equations (for  $\rho$  and  $s$ ). Possible type of the Lagrangians are proposed as

$$L_\phi + L_\psi = - \int_M D_t \phi \rho d^3\mathbf{x} - \int_M D_t \psi \rho s d^3\mathbf{x} \quad (31)$$

$$= \int_M \mathcal{L}_\phi(\mathbf{x}, t) d^3\mathbf{x} + \int_M \mathcal{L}_\psi(\mathbf{x}, t) d^3\mathbf{x}, \quad (32)$$

$$\mathcal{L}_\phi(\mathbf{x}, t) = -\rho D_t \phi, \quad \mathcal{L}_\psi(\mathbf{x}, t) = -\rho s D_t \psi, \quad (33)$$

where  $\phi(\mathbf{x}, t)$  and  $\psi(\mathbf{x}, t)$  are scalar gauge fields associated with the material (the minus signs in  $\mathcal{L}$ 's are a matter of convenience, as will become clear later), and may be called the gauge potentials. This Lagrangian is invariant by the same reasoning as above. The two terms on the right-hand side of (31) would assure local conservation of mass and entropy, respectively. It will be found later that the *equations* of mass conservation and entropy conservation are deduced as the results of variational principle.

Thus the total Lagrangian is defined by  $L_T := L_f + L_\varepsilon + L_\phi + L_\psi$ .

## 4. Variational formulation for flows of an ideal fluid

### 4.1. Action principle

According to the previous section, the full Lagrangian is defined by

$$L_T = L_f + L_\varepsilon + L_\phi + L_\psi = \int_M d^3\mathbf{x} \mathcal{L}_T[\mathbf{u}, \rho, s, \phi, \psi],$$

$$\mathcal{L}_T \equiv \frac{1}{2} \rho \langle \mathbf{u}, \mathbf{u} \rangle - \rho \varepsilon(\rho, s) - \rho D_t \phi - \rho s D_t \psi, \quad (34)$$

where  $\mathbf{u}$ ,  $\rho$  and  $\varepsilon$  are the velocity vector, density and internal energy (per unit mass) of the fluid, and  $\phi(\mathbf{x}, t)$  and  $\psi(\mathbf{x}, t)$  are scalar functions, and  $D_t = \partial_t + \mathbf{u} \cdot \nabla$ . In Section 3.1, we saw that the velocity can be represented as

$$\mathbf{u}(\mathbf{x}, t) = D_t \mathbf{x} = \partial_t \mathbf{X}(\mathbf{a}, t), \quad (35)$$

where  $\mathbf{a}$  is the particle coordinate. This is the representation consistent with local gauge invariance, and  $\mathbf{u}$  is the velocity of a material particle  $\mathbf{a}$ .

The action principle is given by

$$\delta I = \delta \int_{t_0}^{t_1} \int_M dt d^3\mathbf{x} \mathcal{L}_T = 0. \quad (36)$$

Usually, in the variational formulation of the Eulerian representation, the Euler equation of motion is derived under the constraints of the continuity equation and the isentropic equation. In the present

analysis, the variational principle based the gauge principle provides us the continuity equation and the isentropic equation as the result of variations of the Lagrangian  $L_T[\mathbf{u}, \rho, s, \phi, \psi]$  with respect to variations of the gauge potentials  $\phi$  and  $\psi$ . The Euler equation of motion is derived as an integrated form in the present case to be described just below.

It is to be remarked here that we have to take into consideration of a certain thermodynamic property. Namely, the fluid is an ideal fluid in which there is no mechanism of dissipation of kinetic energy into heat. That is, there is no heat production within fluid. By thermodynamics, change of the internal energy  $\varepsilon$  and enthalpy  $h = \varepsilon + p/\rho$  can be expressed in terms of changes of density  $\delta\rho$  and entropy  $\delta s$  as

$$\delta\varepsilon = \left(\frac{\partial\varepsilon}{\partial\rho}\right)_s \delta\rho + \left(\frac{\partial\varepsilon}{\partial s}\right)_\rho \delta s = \frac{p}{\rho^2} \delta\rho + T \delta s, \quad (37)$$

$$\delta h = \frac{1}{\rho} \delta p + T \delta s, \quad (38)$$

where  $(\partial\varepsilon/\partial\rho)_s = p/\rho^2$  and  $(\partial\varepsilon/\partial s)_\rho = T$  with  $p$  the fluid pressure and  $T$  the temperature,  $(\cdot)_s$  denoting the change with  $s$  kept fixed. If there is no heat production, we have  $T \delta s = 0$ . Then,

$$\delta\varepsilon = (\delta\varepsilon)_s = \left(\frac{\partial\varepsilon}{\partial\rho}\right)_s \delta\rho = \frac{p}{\rho^2} \delta\rho, \quad \delta h = \frac{1}{\rho} \delta p. \quad (39)$$

However, by an initial condition, the entropy  $s$  may be a function of  $\mathbf{x}$ :  $s_{t=0} = s(\mathbf{x}, 0)$ .

#### 4.2. Outcomes of variations

Writing  $\mathcal{L}_T$  as

$$\mathcal{L}_T = \mathcal{L}_T[\mathbf{u}, \rho, s, \phi, \psi] \equiv \frac{1}{2} \rho \langle \mathbf{u}, \mathbf{u} \rangle - \rho \varepsilon(\rho, s) - \rho (\partial_t + \mathbf{u} \cdot \nabla) \phi - \rho s (\partial_t + \mathbf{u} \cdot \nabla) \psi, \quad (40)$$

we take variations of the field variables  $\mathbf{u}$ ,  $\rho$ ,  $s$  and potentials  $\phi$  and  $\psi$ . Independent variations are taken for those variables. Substituting the variations  $\mathbf{u} + \delta\mathbf{u}$ ,  $\rho + \delta\rho$ ,  $s + \delta s$ ,  $\phi + \delta\phi$  and  $\psi + \delta\psi$  into  $\mathcal{L}_T[\mathbf{u}, \rho, s, \phi, \psi]$  and writing its variation as  $\delta\mathcal{L}_T$ , we obtain

$$\delta\mathcal{L}_T = \delta\mathbf{u} \cdot \rho (\mathbf{u} - \nabla\phi - s \nabla\psi) - \delta s \rho D_t \psi + \delta\rho \left(\frac{1}{2} u^2 - h - D_t \phi - s D_t \psi\right) \quad (41)$$

$$+ \delta\phi (\partial_t \rho + \nabla \cdot (\rho \mathbf{u})) - \partial_t (\rho \delta\phi) - \nabla \cdot (\rho \mathbf{u} \delta\phi) + \delta\psi (\partial_t (\rho s) + \nabla \cdot (\rho s \mathbf{u})) - \partial_t (\rho s \delta\psi) - \nabla \cdot (\rho s \mathbf{u} \delta\psi), \quad (42)$$

where  $h$  is the specific enthalpy defined by  $h = \varepsilon + \rho (\partial\varepsilon/\partial\rho)_s = \varepsilon + p/\rho$ .

Thus, the variational principle,  $\delta I = 0$  for independent arbitrary variations  $\delta\mathbf{u}$ ,  $\delta\rho$  and  $\delta s$ , results in

$$\delta\mathbf{u} : \mathbf{u} = \nabla\phi + s \nabla\psi, \quad (43)$$

$$\delta\rho : \frac{1}{2} u^2 - h - D_t \phi - s D_t \psi = 0, \quad (44)$$

$$\delta s : D_t \psi \equiv \partial_t \psi + \mathbf{u} \cdot \nabla\psi = 0. \quad (45)$$

Using (43) and (45), we have

$$D_t \phi = \partial_t \phi + \mathbf{u} \cdot \nabla\phi = \partial_t \phi + \mathbf{u} \cdot (\mathbf{u} - s \nabla\psi) = u^2 + \partial_t \phi + s \partial_t \psi.$$

Using this and (45), Eq. (44) can be rewritten as

$$\frac{1}{2}\mathbf{u}^2 + h + \partial_t \phi + s \partial_t \psi = 0. \quad (46)$$

This is regarded as an integral of motion, as interpreted below. From the variations of  $\delta\phi$  and  $\delta\psi$ , we obtain

$$\delta\phi: \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (47)$$

$$\delta\psi: \partial_t (\rho s) + \nabla \cdot (\rho s \mathbf{u}) = 0. \quad (48)$$

Using (47), the second equation can be rewritten as

$$\partial_t s + \mathbf{u} \cdot \nabla s = D_t s = 0. \quad (49)$$

i.e. the motion is *adiabatic*. Thus, the continuity equation (47) and the entropy equation (49) have been derived from the variational principle. These must be supplemented by the equation of particle motion (35) resulting from the gauge invariance of  $L_f$ , while the condition (39) is consistent with (49).

If the heat production within the fluid by dissipation of kinetic energy is to be taken into account, the second term ( $-\delta s \rho D_t \psi$ ) of (41) must be supplemented by an additional term  $-\delta s \rho (\partial \varepsilon / \partial s)_\rho = -\delta s \rho T$  ( $T$  is the thermodynamic temperature). In many traditional approaches of Eulerian variations, this term is retained (e.g. Herivel, 1955; Seliger and Whitham, 1968; Bretherton, 1970; Salmon, 1988, etc.), and this inevitably leads to  $D_t \psi = -(\partial \varepsilon / \partial s)_\rho = -T$ . This is an awkward relation implying that the potential  $\psi$  keeps changing when the temperature is not zero. This has no support from physics. In the present case, we have  $D_t \psi = 0$  and no such problem arises. Owing to this equation, the present solution is equivalent to the classical Clebsch solution (Appendix B; Lamb, 1932, Section 167). With the velocity (43), the vorticity  $\boldsymbol{\omega}$  is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \nabla s \times \nabla \psi. \quad (50)$$

This implies that the vorticity is connected with non-uniformity of entropy.

It is shown in Appendix B that Euler's equation of motion,

$$\partial_t \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} = -\nabla \left( \frac{1}{2} \mathbf{u}^2 + h \right), \quad (51)$$

is satisfied by Eq. (46) together with the definitions (43) and (50) under the conditions (45) and (49), and under the barotropic relation  $h(p) = \int^p dp' / \rho(p')$ . In this case, the helicity vanishes (Appendix B).

Equation (51) can be written also as

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla h, \quad \left( = -\frac{1}{\rho} \text{grad } p \right), \quad (52)$$

because of the identity:  $\boldsymbol{\omega} \times \mathbf{u} = (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \left( \frac{1}{2} \mathbf{u}^2 \right)$ .

### 4.3. Homentropic fluid

For a homentropic fluid in which the entropy  $s$  is a uniform constant  $s_0$  at all points, we have  $e = e(\rho)$ , and

$$d\varepsilon = \frac{p}{\rho^2} d\rho, \quad dh = \frac{1}{\rho} dp. \quad (53)$$

from (37) and (38) since  $\delta s = 0$ . In addition, the motion is *irrotational*. In fact, from (43), we have

$$\mathbf{u} = \nabla \Phi, \quad \Phi = \phi + s_0 \psi, \quad (54)$$

i.e. the velocity field has a potential  $\Phi$ , and  $\omega = 0$  from (50). The integral (46) becomes

$$\frac{1}{2}u^2 + h + \partial_t \Phi = 0. \quad (55)$$

The Euler equation (51) reduces to

$$\partial_t \mathbf{u} + \nabla \left( \frac{1}{2}u^2 \right) = -\nabla h \quad \text{where} \quad \nabla h = \frac{1}{\rho} \nabla p. \quad (56)$$

Note that the left-hand side is the *material* time derivative for the potential velocity  $u^k = \partial_k \Phi$ . In fact, using  $\partial_i(u^2/2) = u^k \partial_i u^k = (\partial_k \Phi) \partial_i \partial_k \Phi = (\partial_k \Phi) \partial_k \partial_i \Phi = u^k \partial_k u^i$ , we obtain

$$\partial_t \mathbf{u} + \nabla \left( \frac{1}{2}u^2 \right) = \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = D_t \mathbf{u}. \quad (57)$$

Thus, as far as the action principle is concerned for a homentropic fluid, Euler's equation of motion reduces to that for *potential* flows of a perfect fluid. In the traditional approaches, this property is thought as a defect of the formulation of Eulerian variation described in the previous section, because the action principle must deduce the equations for rotational flows as well. In order to remove this (apparent) flaw, Lin (1963) introduced the condition for the conservation of the identity of particles denoted by  $\mathbf{a} = (a^k)$ , which is represented by an additional subsidiary Lagrangian of the form  $\int A_k \cdot D_t a^k d^3 \mathbf{x}$ . This introduces three potentials  $A_k(\mathbf{x}, t)$  as a set of Lagrange multipliers of conditional variation, which are considered to be somewhat mysterious or lack a physical significance (Seliger and Whitham, 1968; Bretherton, 1970; Salmon, 1988).

Let us recall that we are considering the Lagrangian  $L_T$  satisfying the symmetry of parallel translation, whereas the flow field has another symmetry of rotational invariance, which was studied in the previous papers (Kambe, 2003a,b). Equation (50) implies that the entropy  $s$  plays the role to identify each fluid particle owing to the entropy equation (49) and that local rotation is captured by the mechanism. However, in a homentropic fluid, there is no such machine to identify each fluid particle. Gauge invariance with respect to local rotation could be a candidate instead of  $s$ . The problem of vorticity in a homentropic fluid is out of the scope of present study, and will be investigated elsewhere in future.

As far as the flow field is characterized by the translational symmetry (only), we have arrived at the present result, i.e. the flow field should be *irrotational* if the fluid is homentropic. The fluid motion is driven by the velocity potential  $\Phi$ , where  $\Phi = \phi + s_0 \psi$ .

It is interesting to recall that the flow of a superfluid in the degenerate ground state is represented by using a velocity potential (Landau and Lifshitz, 1987, Section 137; Schutz and Sorkin, 1977; Lin, 1963). Therefore the corresponding velocity is irrotational (Pethick and Smith, 2002, Chapter 7). In this case, local rotation would not be captured.

## 5. Variations and Noether's theorem

The equation of momentum conservation results from the Noether theorem associated with the local translational symmetry. Variations are taken with respect to translational transformations with the gauge potentials fixed.



The action  $I$  is defined by (34) as

$$I = \int_{t_0}^{t_1} dt \int_M d^3\mathbf{x} [\mathcal{L}_f + \mathcal{L}_\varepsilon + \mathcal{L}_\phi + \mathcal{L}_\psi], \quad \mathcal{L}_f(x) = \frac{1}{2} \rho \langle \mathbf{u}, \mathbf{u} \rangle, \quad (58)$$

where  $\mathcal{L}_\varepsilon = -\rho \varepsilon(\rho, s)$ ,  $\mathcal{L}_\phi = -\rho D_t \phi$  and  $\mathcal{L}_\psi = -\rho s D_t \psi$ .

### 5.1. Local variations

We consider the following infinitesimal coordinate transformation:

$$\mathbf{x}'(\mathbf{x}, t) = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t). \quad (59)$$

By this transformation, a volume element  $d^3\mathbf{x}$  is changed to

$$d^3\mathbf{x}' = J d^3\mathbf{x} = (1 + \partial_k \xi^k) d^3\mathbf{x},$$

where  $J \equiv \partial(x'^1, x'^2, x'^3) / \partial(x^1, x^2, x^3) = 1 + \partial_k \xi^k$  (where  $\partial_k \xi^k = \text{div } \boldsymbol{\xi}$ ) is the Jacobian of the transformation (up to the first order terms). From (18), the velocity  $\mathbf{u}(\mathbf{x})$  is transformed locally as

$$\mathbf{u}'(\mathbf{x}') = \mathbf{u}(\mathbf{x}) + D_t \boldsymbol{\xi},$$

where the points  $\mathbf{x}$  and  $\mathbf{x}'$  are the same points with respect to the inertial space  $F_M$ . We denote this transformation by

$$\Delta \mathbf{u} = \mathbf{u}'(\mathbf{x}') - \mathbf{u}(\mathbf{x}) = D_t \boldsymbol{\xi}. \quad (60)$$

According to the change of volume element  $d^3\mathbf{x}$ , there is change of density  $\rho$ . In view of the invariance of the mass, we have

$$\rho(\mathbf{x}) d^3\mathbf{x}(\mathbf{x}) = \rho'(\mathbf{x}') d^3\mathbf{x}'(\mathbf{x}') \quad \text{thus} \quad \Delta(\rho d^3\mathbf{x}) = 0. \quad (61)$$

Hence, we obtain  $\rho(\mathbf{x}) = (1 + \text{div } \boldsymbol{\xi}) \rho'(\mathbf{x}')$ . Therefore,

$$\Delta \rho = \rho'(\mathbf{x}') - \rho(\mathbf{x}) = -\rho \text{div } \boldsymbol{\xi} = -\rho \partial_k \xi^k, \quad (62)$$

to the first order of  $|\boldsymbol{\xi}|$ . The invariance of entropy  $s \rho d^3\mathbf{x}(\mathbf{x}) = s' \rho' d^3\mathbf{x}'(\mathbf{x}')$  results in

$$\Delta s = s'(\mathbf{x}') - s(\mathbf{x}) = 0. \quad (63)$$

The gauge fields  $D_t \phi$  and  $D_t \psi$  remain unvaried

$$\Delta(D_t \phi) = 0, \quad \Delta(D_t \psi) = 0.$$

Combining with (61), we obtain

$$\Delta(\mathcal{L}_\phi d^3\mathbf{x}) = 0, \quad \Delta(\mathcal{L}_\psi d^3\mathbf{x}) = 0. \quad (64)$$

The variation field  $\boldsymbol{\xi}(\mathbf{x}, t)$  is constrained so as to vanish on the boundary surface  $S$  of  $M \subset E^3$ , as well as at both ends of time  $t_0$  and  $t_1$  for the action  $I$  (where  $M$  is chosen arbitrarily):

$$\boldsymbol{\xi}(\mathbf{x}_S, t) = 0 \quad \text{for any } t \text{ for } \mathbf{x}_S \in S = \partial M, \quad (65)$$

$$\boldsymbol{\xi}(\mathbf{x}, t_0) = 0, \quad \boldsymbol{\xi}(\mathbf{x}, t_1) = 0 \quad \text{for } \forall \mathbf{x} \in M. \quad (66)$$

When we consider the symmetry with respect to global transformation, we take the limit:

$$\xi(\mathbf{x}, t) \rightarrow \xi_0 \quad (\text{a uniform constant vector}).$$

### 5.2. Invariant variation

It is required that, under the infinitesimal variations (59)–(64), the action  $I$  should be invariant, i.e.

$$0 = \Delta I \equiv I' - I = \int dt \int_{M'} d^3\mathbf{x}' [\mathcal{L}_f(\mathbf{u}', \rho') + \mathcal{L}_\varepsilon(\rho', s') + \mathcal{L}'_\phi + \mathcal{L}'_\psi](\mathbf{x}', t) \\ - \int dt \int_M d^3\mathbf{x} [\mathcal{L}_f(\mathbf{u}, \rho) + \mathcal{L}_\varepsilon(\rho, s) + \mathcal{L}_\phi + \mathcal{L}_\psi](\mathbf{x}, t).$$

The integration space  $M'$  is the same as  $M$  with respect to the frame  $F_M$ , although different expressions are given. Using the original variables in the first integral, we have

$$\Delta I = \int dt \int_M d^3\mathbf{x} J (\mathcal{L}_f(\mathbf{u} + \Delta\mathbf{u}, \rho + \Delta\rho) + \mathcal{L}_\varepsilon(\rho + \Delta\rho, s + \Delta s)) \\ - \int dt \int_M d^3\mathbf{x} [\mathcal{L}_f(\mathbf{u}, \rho) + \mathcal{L}_\varepsilon(\rho, s)] = 0, \quad (67)$$

by using (64), where  $J d^3x = (1 + \partial_k \zeta^k) d^3x$ . Thus we obtain

$$\Delta I = \int dt \int_M d^3\mathbf{x} \left\{ \frac{\partial \mathcal{L}_f}{\partial \mathbf{u}} \Delta\mathbf{u} + \frac{\partial \mathcal{L}_f}{\partial \rho} \Delta\rho + \left( \frac{\partial \mathcal{L}_\varepsilon}{\partial \rho} \right)_s \Delta\rho + \left( \frac{\partial \mathcal{L}_\varepsilon}{\partial s} \right)_\rho \Delta s \right. \\ \left. + [\mathcal{L}_f(\mathbf{u}, \rho) + \mathcal{L}_\varepsilon(\rho, s)] \partial_k \zeta^k \right\},$$

to the first order of variations. These include all the terms associated with the  $\xi$ -variation. Substituting (60), (62) and (63), we have

$$\Delta I = \int dt \int_M d^3\mathbf{x} \left\{ \frac{\partial \mathcal{L}_f}{\partial \mathbf{u}} D_t \xi + \left( \frac{\partial \mathcal{L}_f}{\partial \rho} + \left( \frac{\partial \mathcal{L}_\varepsilon}{\partial \rho} \right)_s \right) (-\rho \partial_k \zeta^k) + [\mathcal{L}_f + \mathcal{L}_\varepsilon] \partial_k \zeta^k \right\} = 0, \quad (68)$$

where  $\mathcal{L}_f + \mathcal{L}_\varepsilon = \frac{1}{2} \rho u^2 - \rho \varepsilon$ , and

$$\frac{\partial \mathcal{L}_f}{\partial \mathbf{u}} = \rho \mathbf{u}, \quad \frac{\partial \mathcal{L}_f}{\partial \rho} = \frac{1}{2} u^2, \quad \left( \frac{\partial \mathcal{L}_\varepsilon}{\partial \rho} \right)_s = -h. \quad (69)$$

It is immediately seen that the second and third terms of (68) can be combined

$$-\rho \left( \frac{\partial \mathcal{L}_f}{\partial \rho} + \left( \frac{\partial \mathcal{L}_\varepsilon}{\partial \rho} \right)_s \right) (\partial_k \zeta^k) + [\mathcal{L}_f + \mathcal{L}_\varepsilon] \partial_k \zeta^k \\ = \rho \left( -\frac{1}{2} u^2 + h + \frac{1}{2} u^2 - e \right) \partial_k \zeta^k \\ = \rho (h - e) \partial_k \zeta^k = \rho \frac{P}{\rho} \partial_k \zeta^k = p \partial_k \zeta^k, \quad (70)$$

from (69) and  $h = \varepsilon + p/\rho$ . Hence, the second and third terms of (68) are reduced to the single term  $p(\partial_k \xi^k)$ , which can be expressed further as  $\partial_k(p \xi^k) - (\partial_k p) \xi^k$ .

### 5.3. Noether's theorem

We now consider the outcome obtained from the arbitrary variation of  $\xi$ . We write (68) as

$$\Delta I = \int dt \int_M d^3x F[\xi^k, \partial_\alpha \xi^k] = 0. \tag{71}$$

By using (70), the integrand  $F[\xi^k, \partial_\alpha \xi^k]$  reduces to

$$F[\xi^k, \partial_\alpha \xi^k] = \frac{\partial \mathcal{L}_f}{\partial \mathbf{u}} \mathbf{D}_t \xi + p(\partial_k \xi^k). \tag{72}$$

In view of the definitions  $\mathbf{D}_t \xi^k = \partial_t \xi^k + u^l \partial_l \xi^k$ , this can be rewritten as

$$F[\xi^k, \partial_\alpha \xi^k] = \xi^k \left[ -\partial_t \left( \frac{\partial \mathcal{L}_f}{\partial u^k} \right) - \partial_l \left( u^l \frac{\partial \mathcal{L}_f}{\partial u^k} \right) - (\partial_k p) \right] + \text{Div}, \tag{73}$$

where the divergence terms are collected in the term Div:

$$\text{Div} = \partial_t \left( \frac{\partial \mathcal{L}_f}{\partial u^k} \xi^k \right) + \partial_l \left( u^l \frac{\partial \mathcal{L}_f}{\partial u^k} \xi^k \right) + \partial_k (p \xi^k). \tag{74}$$

Using (69), the expression (73) with (74) becomes

$$F[\xi^k, \partial_\alpha \xi^k] = \xi^k [-\partial_t(\rho u^k) - \partial_l(\rho u^l u^k) - \partial_k p] + \partial_t(\rho u^k \xi^k) + \partial_l(u^l \rho u^k \xi^k) + \partial_l(p \delta_k^l \xi^k). \tag{75}$$

Substituting (75) into (71), the variational principle (71) can be written as

$$\begin{aligned} \Delta I = & \int dt \int_M d^3x \xi^k (-\partial_t(\rho u^k) - \partial_l(\rho u^l u^k) - \partial_k p) \\ & + \int dt \left[ \frac{d}{dt} \int_M d^3x (\rho u^k \xi^k) + \int dt \int_S dS n^l (\rho u^l u^k + p \delta_k^l) \xi^k \right] = 0, \end{aligned} \tag{76}$$

where  $S$  is the boundary surface of  $M$  and  $(n^l) = \mathbf{n}$  is a unit outward normal to  $S$ . The terms on the second line are integrated terms, which came from the last three terms of (75). These vanish owing to the imposed conditions (65) and (66).

Thus, the invariance of  $I$  for arbitrary variation of  $\xi^k$  satisfying the conditions (65) and (66) results in

$$\partial_t(\rho u^k) + \partial_l(\rho u^l u^k + p \delta_{lk}) = 0. \tag{77}$$

This is the *conservation equation of momentum*. If we use the continuity equation (47), we obtain the Euler equation of motion:

$$\partial_t u^k + u^l \partial_l u^k + \frac{1}{\rho} \partial_k p = 0. \tag{78}$$

which is equivalent to (52).

Now, we can consider the outcome of global gauge invariance with respect to a global translation of  $\zeta^k = \text{const}$ , without the conditions (65) and (66). Using Eq. (77) obtained from the variational principle (described above), the first line of (76) vanishes. Thus, for  $\zeta^k = \text{const}$ , we obtain from (76),

$$\zeta^k \left[ \frac{d}{dt} \int_M d^3x (\rho u^k) + \int_S dS n^l (\rho u^l u^k + p \delta_k^l) \right] = 0 \quad (79)$$

taking the constant  $\zeta^k$  out of the integral signs. For arbitrary  $\zeta^k$  ( $k = 1, \dots, 2, 3$ ), the expression within [ ] must vanish. Therefore,

$$\frac{d}{dt} \int_M d^3x (\rho u^k) = - \int_S dS n^l (\rho u^l u^k + p \delta_k^l), \quad (80)$$

for  $k=1, 2, 3$ . This states conservation of total momentum. Namely, rate of change of the  $k$ th component of the total momentum  $\int_M d^3x \rho u^k$  is given by the influx of momentum from outside of  $S$ ,  $-\int_S dS n^l \rho u^l u^k$  and rate of increase of momentum within  $M$  by the pressure force  $-\int_S dS n^k p$  on the surface  $S$  from outside.

## 6. Summary and discussions

Following the scenario of the gauge principle in the field theory of physics, it is found that the variational principle of fluid motions can be reformulated successfully in terms of covariant derivative and Lagrangians, where the Lagrangians are determined such that a gauge invariance is satisfied under translational transformations, i.e. local Galilean transformations. In order to consider *local* gauge-invariance, an indispensable element is the existence of a background fluid *material*, which is characterized thermodynamically by mass density and entropy (per unit mass).

The covariant derivative is an essential building block of the gauge theory. According to the gauge principle, a gauge-covariant derivative  $D_t$  with respect to time  $t$  is defined by introducing a gauge term. Galilean invariance requires that the covariant derivative should be the *convective* time derivative following the motion of background material, i.e. the so-called Lagrange derivative.

Using the gauge-covariant operator  $D_t$ , a free-field Lagrangian  $L_f$  and Lagrangians  $L_\phi$  and  $L_\psi$  (associated with the gauge fields) are defined under the gauge symmetry of parallel translation. The Lagrangians  $L_\phi$  and  $L_\psi$  include gauge potentials (two scalar functions  $\phi$  and  $\psi$ ) in the form of  $D_t \phi$  and  $D_t \psi$ , respectively. From the variational principle, i.e. the action principle, an equation of motion is derived, which is an integrated form of the Euler equation. In addition, the equation of continuity and equation of entropy conservation are derived simultaneously from variations of  $L_\phi$  and  $L_\psi$ . With this formulation, it is seen now that there is close analogy between Fluid Mechanics and Theory of Electromagnetism.

In the conventional variational formulations, Euler's equation of motion is derived under the constraints of the continuity equation and the isentropic flow, whereas the present analysis provides us both of the equations as outcome of variations with respect to the gauge potentials  $\phi$  and  $\psi$ . It may seem that the gauge potentials play similar role in the variation to the role of Lagrange multipliers in the conventional variation of Eulerian representation. But the gauge potentials have intrinsic physical significance in the framework of the gauge theory, while the Lagrange multipliers are used to impose known conditions on the system under consideration. In this sense, both are absolutely different in nature.

It is found that a general solution obtained in the present formulation is equivalent to the classical Clebsch solution. If entropy of the fluid is non-uniform, the flow will be rotational. However, if the entropy is uniform throughout the space (i.e. homentropic), then the flow field reduces to that of a potential flow. In this regard, Bretherton (1970) proposed a Hamilton's principle for ideal fluids, which is a variational approach from Lagrangian particle aspect. However, in this formulation, both the equation of continuity and the equation of entropy conservation are identities built into the geometrical specification of the system (according to the interpretation in the paper). In the Bretherton's procedure, the equation of momentum conservation is derived finally. However, it is not clear on what principle the variational transformations of density and velocity are based. In the present formulation, it is the gauge transformation, i.e. local Galilean transformation.

At present, we have the expectation that the vorticity can be handled properly by a Lagrangian which takes into account rotational symmetry appropriately. The rotational symmetry was studied in the previous papers (Kambe, 2003a,b). The velocity field derived from such a Lagrangian would support vorticity field in general. The problem of vorticity in a homentropic fluid is out of the scope of present study, and will be investigated elsewhere in future.

It is noteworthy that the flow of a superfluid in the degenerate ground state is represented by using a velocity potential (Landau and Lifshitz, 1987, Section 137; Lin, 1963). From recent advance of studies of the Bose–Einstein condensation, it becomes increasingly adequate that such a fluid of macroscopic number of bosons is represented by a single wave function in the degenerate ground state, where the quantum-mechanical current is described by a potential function (phase of the wave function). Therefore the corresponding velocity is irrotational (Pethick and Smith, 2002, Chapter 7). In this case, local rotation would not be captured and resulting flow would be irrotational (Schutz and Sorkin, 1977).

From gauge invariance of the Lagrangian with respect to translational transformations, a differential conservation equation of momentum has been deduced as Noether's theorem.

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## Appendix A. Galilean-invariant Lagrangian

The Galilean transformation is regarded as a limiting case of the Lorentz transformation of space–time ( $x^\mu = (t, \mathbf{x})$ ) as  $v/c \rightarrow 0$ . The Lorentz invariant Lagrangian  $A_L^{(0)}$  in the limit as  $v/c \rightarrow 0$ , is defined by

$$A_L^{(0)} dt = \int_M d^3x \rho(x) \left( \frac{1}{2} \langle v(x), v(x) \rangle - \varepsilon - c^2 \right) dt \quad (\text{A.1})$$

(Landau and Lifshitz, 1987, Section 133). The third term  $-c^2 dt$  is not only necessary, but indispensable, so as to satisfy the Lorentz-invariance (Landau and Lifshitz, 1975, Section 87). However, this term gives a constant term  $c^2 \mathcal{M} dt$  for flows in a finite domain  $M \subset E^3$ , where  $\mathcal{M} = \int d^3x \rho(x)$  is the total mass in

the domain  $M$ . In carrying out variations, the total mass  $\mathcal{M}$  is fixed to a constant. Only when we need to consider the Galilean invariance, we must use this Lagrangian  $A_L^{(0)}$ , rather than the  $L_F$  of (28).

## Appendix B. Clebsch solution

According to Lamb (1932, Section 167), the Euler equation of motion,

$$\partial_t \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} = -\nabla \left( \frac{1}{2} v^2 + \int^x \frac{dp}{\rho} \right), \quad (\text{B.1})$$

with  $p = p(\rho)$ , can be solved in general by

$$\mathbf{v} = \nabla \phi + \lambda \nabla \psi, \quad (\text{B.2})$$

$$\frac{1}{2} v^2 + h + \partial_t \phi + \lambda \partial_t \psi = 0, \quad h = \int \frac{dp}{\rho}, \quad (\text{B.3})$$

$$D_t \lambda = 0, \quad D_t \psi = 0, \quad D_t = \partial_t + \mathbf{v} \cdot \nabla, \quad (\text{B.4})$$

where  $\lambda$ ,  $\psi$ ,  $\psi$  are scalar functions of  $\mathbf{x}$ . The continuity equation provides an equation for the potential  $\phi$ . This solution represents the vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{v}$  in the form,

$$\boldsymbol{\omega} = \nabla \lambda \times \nabla \psi. \quad (\text{B.5})$$

In fact, using (B.2) and (B.5), we have

$$\partial_t \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} = \nabla (\partial_t \phi + \lambda \partial_t \psi) + (D_t \lambda) \nabla \psi - (D_t \psi) \nabla \lambda,$$

where the last two terms vanish due to (B.4). Thus, Eq. (B.1) implies (B.3) where integration constant can be absorbed in the function  $\phi$ . The vortex lines are the intersections of the families of surfaces  $\lambda = \text{const}$  and  $\psi = \text{const}$ . These surfaces are moving with the fluid by (B.4). If the scalar product  $\boldsymbol{\omega} \cdot \mathbf{v}$  is integrated over a volume  $V$  including a number of closed vortex filaments, the helicity  $H[V]$  vanishes

$$H[V] \equiv \int_V \boldsymbol{\omega} \cdot \mathbf{v} d^3 \mathbf{x} = \int_V (\nabla \lambda \times \nabla \psi) \cdot \nabla \phi d^3 \mathbf{x} = \int_V \nabla \cdot [\phi \boldsymbol{\omega}] d^3 \mathbf{x} = 0$$

(Bretherton, 1970). For a general velocity field, the helicity  $H$  is a measure of knottedness of the vortex lines and does not vanish in general.

## Appendix C. Scale invariance

Suppose that we have an action functional defined by

$$I = \int \int dt d^3 \mathbf{x} \mathcal{L}(t, \mathbf{x}), \quad \mathcal{L} = \frac{1}{2} \rho u^2 - \rho \varepsilon(\rho) - \rho D_t \phi, \quad (\text{C.1})$$

where the velocity  $\mathbf{u}$  is irrotational and represented by  $\mathbf{u} = \nabla \phi$  and  $\varepsilon$  is the internal energy per unit mass. This is the action for flows of a fluid of uniform entropy with the velocity potential  $\phi$  and in this case  $\varepsilon$  is a function of density  $\rho$  only (Section 4.3). Consider the following scaling transformation:

$$t \rightarrow t' = e^{2\lambda} t, \quad \mathbf{x} \rightarrow \mathbf{x}' = e^\lambda \mathbf{x}, \quad (\text{C.2})$$

where  $\lambda$  is a parameter of transformation. Under this, the fields transform as

$$\begin{aligned} \rho &\rightarrow \rho' = e^{-3\lambda}\rho, & \varepsilon &\rightarrow \varepsilon' = e^{-2\lambda}\varepsilon, & h &\rightarrow h' = e^{-2\lambda}h, \\ \mathbf{u} &\rightarrow \mathbf{u}' = e^{-\lambda}\mathbf{u}, & \phi &\rightarrow \phi' = \phi, & \mathbf{D}_t &\rightarrow \mathbf{D}'_t = e^{-2\lambda}\mathbf{D}_t, \end{aligned}$$

where  $\mathbf{D} = \partial_t + \mathbf{u} \cdot \nabla$ . It is immediately checked that the action is invariant by this scaling transformation:  $\mathcal{L} = \mathcal{L}'$ .

Using the relation that  $\mathbf{D}_t \phi = \partial_t \phi + \mathbf{u} \cdot \nabla \phi = \partial_t \phi + u^2$ , the Lagrangian density  $\mathcal{L}$  can be written as

$$\mathcal{L} = -\rho \partial_t \phi - \mathcal{H}, \quad \mathcal{H} \equiv \frac{1}{2} \rho u^2 + V(\rho), \quad V = \rho \varepsilon, \tag{C.3}$$

where  $\mathcal{H}$  is a Hamiltonian density. Integrating by part, this is rewritten as

$$\mathcal{L} = \phi \rho_t - \mathcal{H},$$

where  $\rho_t = \partial_t \rho$  and the term  $\partial_t(\phi \rho)$  is omitted in view of the time integral of (C.1). This implies that  $\phi$  and  $\rho$  are mutually-conjugate canonical variables with  $\rho$  a generalized coordinate and  $\phi$  its associated momentum.

Writing (C.2) as  $x'^\mu = X^\mu(x, \lambda)$  with  $t = x^0$  and  $\mathbf{x} = (x^1, x^2, x^3)$ , the Noether theorem is given by the conservation equation  $\partial_\mu \mathcal{J}^\mu = 0$ , where the conserved current  $\mathcal{J}^\mu$  ( $\mu = 0, 1, 2, 3$ ) is defined by

$$\mathcal{J}^\mu = \mathcal{L} \frac{\partial X^\mu}{\partial \lambda} \Big|_{\lambda=0} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \rho)} \left[ \frac{\partial \rho'}{\partial \lambda} - \partial_\nu \rho \frac{\partial X^\nu}{\partial \lambda} \right]_{\lambda=0}$$

(Soper, 1976, Section 9.1). Using the explicit form of  $X^\mu$ ,  $\mathcal{L}$  and  $\mathcal{H}$  given above, we obtain  $\partial X^\mu / \partial \lambda = x^\mu$  and  $\partial \rho' / \partial \lambda = -3\rho$  at  $\lambda = 0$ . The current is given by  $\mathcal{J}^k = x^k \mathcal{L}$  ( $k = 1, 2, 3$ ) and

$$\mathcal{J}^0 = 2t \mathcal{L} + \phi (-3\rho - 2t \rho_t - x^k \partial_k \rho) = -2t \mathcal{H} + x^k \rho u^k - \partial_k (x^k \rho \phi),$$

where  $\partial \mathcal{L} / \partial(\partial_\mu \rho) = \phi$ . Thus, the conserved quantity is given by

$$\begin{aligned} J^0 &= -2t H + \int x^k \rho u^k d^3 \mathbf{x}, \\ H &= \int \left( \frac{1}{2} \rho u^2 + V(\rho) \right) d^3 \mathbf{x}, \quad V(\rho) = \rho \varepsilon(\rho) = \int^\rho h(\rho') d\rho', \end{aligned} \tag{C.4}$$

where  $H$  is the total energy which is also conserved. In fact, we have

$$\partial_t H = \int \left( \frac{1}{2} u^2 + h \right) \partial_t \rho d^3 \mathbf{x} + \int \rho u^k \partial_t u^k = 0.$$

This can be verified by using the continuity equation (47) and the equation of motion (52). Similarly, we have the invariance of  $J^0$ ,

$$\partial_t J^0 = -2H + \int x^k \partial_t (\rho u^k) d^3 \mathbf{x} = 0,$$

for a monoatomic gas of thermodynamic property  $V = c\rho^\gamma / (\gamma - 1)$  and  $p = c\rho^\gamma$  with  $\gamma = \frac{5}{3}$  ( $c$ : a constant), by using the momentum conservation equation (77). See Jackiw (2002) for the conserved quantities of the Chaplygin gas.



Another example of scale invariance is given by Newtonian gravitational interaction in which the interaction potential is described by

$$V = G \int \frac{\rho(\mathbf{x}) \rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \quad G: \text{ gravitation constant.}$$

The scale invariance is obtained if the transformation law is as follows:

$$\begin{aligned} \mathbf{x} &\rightarrow \mathbf{x}' = e^{\lambda} \mathbf{x}, & t &\rightarrow t' = e^{(3/2)\lambda} t, \\ \mathbf{u} &\rightarrow \mathbf{u}' = e^{-(1/2)\lambda} \mathbf{u}, & \rho &\rightarrow \rho' = e^{-3\lambda} \rho. \end{aligned}$$

This implies Kepler's third law.

## References

- Aitchison, I.J.R., Hey, A.J.G., 1982. *Gauge Theories in Particle Physics*. Adams Hilger, Bristol.
- Arnold, V.I., 1966/1978. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)* 16, 319–361; *Mathematical Methods of Classical Physics*, Appendix 2. Springer, Berlin.
- Bretherton, F.P., 1970. A note on Hamilton's principle for perfect fluids. *J. Fluid Mech.* 44, 19–31.
- Frankel, T., 1997. *The Geometry of Physics an Introduction*. Cambridge University Press, Cambridge.
- Herivel, J.W., 1955. The derivation of the equations of motion of an ideal fluid by Hamilton's principle. *Proc. Cambridge Phil. Soc.* 51, 344–349.
- Jackiw, R., 2002. *Lectures on Fluid Dynamics: A Particle Theorist's View of Supersymmetric, Non-Abelian, Noncommutative Fluid Mechanics and D-Branes*. Springer, Berlin.
- Kambe, T., 2003a. Gauge principle for flows of a perfect fluid. *Fluid Dyn. Res.* 32, 193–199.
- Kambe, T., 2003b. Gauge principle and variational formulation for flows of an ideal fluid. *Acta Mech. Sin.* 19, 437–452.
- Kambe, T., 2004. *Geometrical Theory of Dynamical Systems and Fluid Flows*. World Scientific Publishing, Singapore.
- Lamb, H., 1932. *Hydrodynamics*. Cambridge University Press, Cambridge.
- Landau, L.D., Lifshitz, E.M., 1975. *The Classical Theory of Fields*. fourth ed. Pergamon Press, New York.
- Landau, L.D., Lifshitz, E.M., 1976. *Mechanics*. third ed. Pergamon Press, New York.
- Landau, L.D., Lifshitz, E.M., 1987. *Fluid Mechanics*. second ed. Pergamon Press, New York.
- Lin, C.C., 1963. Hydrodynamics of helium II, in *Liquid Helium*. *Proc. Int. Sch. Phys. Enrico Fermi XXI*. Academic Press, NY, pp. 93–146.
- Marsden, J.E., Hughes, T.J.R., 1993. *Mathematical Foundations of Elasticity*. Dover, New York.
- Maugin, G.A., 1993. *Material Inhomogeneities in Elasticity*. Chapman & Hall, London.
- Noether, E., 1918. Invariante Variationsproblem. *Kgl.-Ges. Wiss. Nach. Göttingen, Math. Physik Kl.* 2, 235.
- Pethick, C.J., Smith, H., 2002. *Bose–Einstein condensation*. Cambridge University Press, Cambridge.
- Salmon, R., 1988. Hamiltonian fluid mechanics. *Ann. Rev. Fluid Mech.* 20, 225–256.
- Schutz, B.F., Sorkin, R., 1977. Variational aspects of relativistic field theories, with application to perfect fluids. *Ann. Phys.* 107, 1–43.
- Seliger, R.L., Whitham, G.B., 1968. Variational principle in continuum mechanics. *Proc. Roy. Soc. A* 305, 1–25.
- Serrin, J., 1959. *Mathematical principles of classical fluid mechanics*. In: Flüggé, Truesdell, C. (Eds.), *Encyclopedia of Physics*. Springer, Berlin, pp. 125–263.
- Soper, D.E., 1976. *Classical Field Theory*. Wiley, New York.
- Utiyama, R., 1978. *Theory of General Relativity* [in Japanese], Syokabo (Tokyo, Japan); Utiyama R 1956. Invariant theoretical interpretation of interaction. *Phys. Rev. A* 101, 1597–1607.
- Weinberg, S., 1995/1996. *The Quantum Theory of Fields*. vols. I and II. Cambridge University Press, Cambridge.